DUAL COMPLEX OF CY PAIRS

Let (X, Δ) dlt pair

$D(\Delta)$ is CW-complex such that

0-cells $\longleftrightarrow$ irred. comp. of Δ
Δ_i

1-cells $\longleftrightarrow$ $\Delta_i \cap \Delta_j$

$\vdots$

k-cells $\longleftrightarrow$ STRATA of Δ
= LC CENTERS of (X, Δ)
= irr. comp. of
$\Delta_{i_0} \cap \dots \cap \Delta_{i_k}$

<u>Conj / Question</u> [Kollár - Xu]
(algebraic version
of Poincaré conj)

Let (X, Δ) be dlt pair
log Calabi-Yau,
i.e. $K_X + \Delta \sim_{\mathbb{Q}} 0$

Then $D(\Delta)$ is PL-homeomorphic
to a sphere or a finite quotient
of a sphere.

Ex. X = smooth proj toric variety

Δ = toric boundary

(X, Δ) snc log CY

Cone $D(\Delta)$ = fan of $X \simeq \mathbb{R}^{n+1}$

where $\dim X = n+1$

$D(\Delta) \simeq S^n$

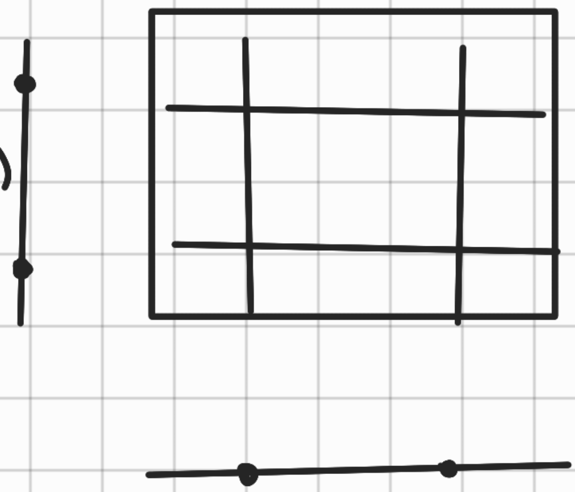
Ex. (X_1, Δ_1) and (X_2, Δ_2) dlt ℓCY pair

$$D(X_1 \times X_2, pr_1^* \Delta_1 + pr_2^* \Delta_2) =$$

$$D(X_1, \Delta_1) \ast D(X_2, \Delta_2)$$

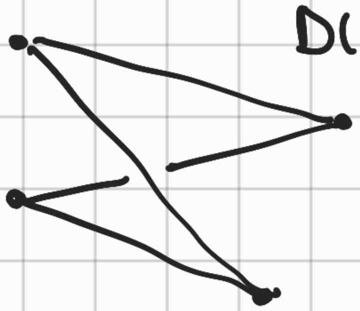
$\mathbb{P}^1 \times \mathbb{P}^1$

$(\mathbb{P}^1, 2pt)$



$(\mathbb{P}^1, 2pt)$

$D(2pt)$ $D(pt)$



$\cong$

S^1

$$S^n \ast S^m = S^{n+m+1}$$

Motivation

- theory of sing.
- mirror symmetry
- $P = W$ conjecture

(see intro of my PhD thesis)

$$\begin{array}{ccc} X_0 \subset X \\ \downarrow \pi \\ 0 \in C \end{array}$$

family of CY varieties over the germ of a curve $(C, 0)$

- X_t is smooth CY proj variety, i.e. $K_{X_t} \sim_{\mathbb{Q}} 0$

[Fujino]
- semistable, i.e. $X_0 = \pi^{-1}(0)$ is reduced
- minimal, i.e. $K_X \sim_{\mathbb{Q}} 0$
- dlt, i.e. (X, X_0) dlt
- maximality, i.e. $\exists$ 0-dim stratum

SYZ conjecture: $\exists\ \varphi : X_t \longrightarrow B$ continuous map with special Lagrangian fibers w.r.t. the CY metric.

Conj. [Konsevitch-Saibelman] $B \simeq D(X_0)$

[Yang Li] (modulo a conj in non-archim. geom) close to a 0-dim stratum of X_0

$$(X, X_0) \simeq_{loc} (\mathbb{C}^{n+1}, x_0 \cdot x_1 \cdot \ldots \cdot x_n = 0)$$

$$X_t = \{ \prod x_i = \varepsilon \} \simeq (\mathbb{C}^\times)^n$$

$$\varphi : X_t \simeq (\mathbb{C}^\times)^n \subset \mathbb{C}^{n+1} \longrightarrow \mathbb{R}^{n+1}$$
$$x_i \longrightarrow \log|x_i|$$

with image a standard simplex i.e. $D(\prod x_i = 0)$

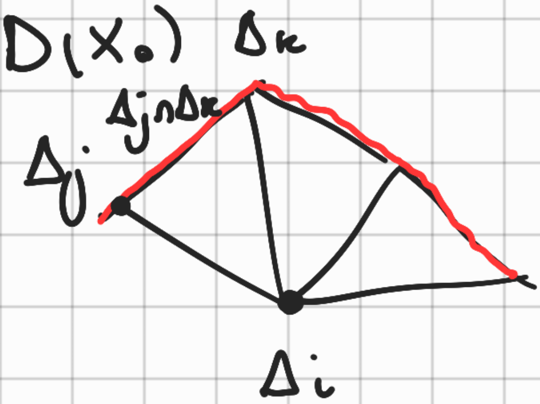
Relation with K-X conj?
$(X, X_0 = \sum \Delta_i)$ dlt $\log CY$

$(\Delta_i, \mathrm{Diff}_{\Delta_i} X_0)$ dlt $\log CY$, since

$$K_{\Delta_i} + \mathrm{Diff}_{\Delta_i} X_0 \sim_{\mathbb{Q}} (\underset{\overset{S}{0}\,\mathbb{Q}}{K_X} + X_0)|_{\Delta_i} \sim_{\mathbb{Q}} 0$$

↑ princ. divisor

$D(X_0)$



neib. of Δ_i in $D(X_0)$ = cone over the red region, i.e. the link of Δ_i.

Link of Δ_i = $D(\mathrm{Diff}_{\Delta_i} \cancel{X_0})$

If $D(\mathrm{Diff}_{\Delta_i} X_0)$ is sphere,

then $D(X_0)$ at the point Δ_i

is a manifold.

C<u>onj</u>. [Matsushita]

If X_t is hyperkähler, i.e.

$\exists\ \sigma \in H^{2,0}(X_t)$ non-deg.

s.t. $\langle \sigma \rangle = H^{*,0}(X_t)$

+ $\pi_1(X_t) = 1$, and if

$f : X_t \to B$ is an <u>alg.</u> fiber

space with $\dim B < \dim X_t$,

then B is smooth

(Hwang : $B \overset{\Downarrow \text{ Thm.}}{\simeq} \mathbb{P}^{\dim X_t/2}$)

[see Huybrechts - Xu , dim 4
Huybrechts - M. , Lagrangian fibration]

STATE OF ART:

[K-X] Conj holds for $\dim X \leq 4$ and $\dim X \leq 5$ if (X, Δ) is snc

[M.] Conj holds if $f: X \to Z$ Mori fiber space with $\rho(X) \leq 2$ or $\dim Z \leq 2$

[Kollár, Simpson, H.-Mazzon-Stevenson, Szabó] geometric $P = W$ conj: conj holds in special interesting varieties i.e. character varieties

§ RATIONAL COHOMOLOGY OF DUAL COMPLEX

Let $(X, \Delta = \sum_{i \in I} \Delta_i)$ snc pair

Goal Find a relation between

$$H^i(\mathcal{O}_\Delta) \qquad\qquad H^i(\mathcal{D}(\Delta))$$

there exists a resolution of

$$\mathcal{O}_\Delta \hookrightarrow \bigoplus \mathcal{O}_{\Delta_i} \to \bigoplus \mathcal{O}_{\Delta_{ij}} \to \dots$$

$$\dots \to \bigoplus_{|J|=q+1} \mathcal{O}_{\Delta_J} \to \qquad J \subset I$$

ex. if Δ is a curve

$$0 \to \mathcal{O}_\Delta \to \underset{\nu_*\mathcal{O}_{\Delta^\nu}}{\underset{\|}{\bigoplus \mathcal{O}_{\Delta_i}}} \to \underset{\text{conductor}}{\bigoplus \mathcal{O}_{\Delta_{ij}}} \to 0$$

and there exists a spectral sequence

$$E_1^{p,q} = \bigoplus_{|J|=q+1} H^p(\mathcal{O}_{\Delta_J}) \Rightarrow H^{p+q}(\mathcal{O}_\Delta)$$

$$\begin{array}{c|ccc}
q & & & \\
 & \bigoplus_{|J|=q+1} H^0(\mathcal{O}_{\Delta_J}) = k^{\#\text{ simplex of dim } q \text{ in } D(\Delta)} & 0 & 0 \\
 & \vdots & & \\
 & \uparrow & & \\
 & \bigoplus H^0(\mathcal{O}_{\Delta_{ij}}) & 0 & 0 \\
 & \uparrow & & \\
 & \bigoplus H^0(\mathcal{O}_{\Delta_1}) & 0 & 0 \\
\hline
 & & & p
\end{array}$$

$\uparrow$

cellular cochain complex for $D(\Delta)$

$H^i(\mathcal{O}_{\Delta_J})$ with $i > 0$

Assumption :

- char $k = 0$, $k = \mathbb{C}$
- log CY
- pair has coregularity 0, i.e. $\dim D(\Delta) = \dim \Delta$,

i.e. $\exists$ 0-dim stratum

Thesis $H^i(\mathcal{O}_{\Delta_J}) = 0 \quad i > 0$.

<u>Sketch.</u> (X, Δ) has cor. zero

$\Downarrow$

$(\Delta_J, \mathrm{Diff}_{\Delta_J} \Delta)$ has cor. zero

$\Downarrow$

Δ_J rationally connected. $\Leftrightarrow Z = pt$

$$\begin{array}{c} (\Delta_J, \overset{D}{\overset{\|}{\mathrm{Diff}_{\Delta_J} \Delta}}) \\ \Big\downarrow \text{ MRC fibration} \\ Z \end{array}$$

the strata of $\mathrm{Diff}_{\Delta_J} \Delta$ dominates Z. $\Rightarrow$
Since $\mathrm{Diff}_{\Delta_J} \Delta$ has a 0-dim stratum, then Z is a pt and Δ_J is rationally conn.

$$\begin{array}{ccc} D & \subset & \Delta_J \\ \downarrow & & \downarrow \pi \\ \pi(D) & \subset & Z \end{array}$$

$\kappa \log \pi(D)$ is a divisor in Z

$$K_{\Delta_J} + D = \pi^*(K_Z + B + J)$$

Let C be a general c.i. curve in X

$$(K_{\Delta_J} + D)\cdot C = 0$$

$$= \begin{cases} \pi^* K_Z \cdot C = K_Z \cdot \pi(C) \geq 0 \\ \pi^* J \cdot C = J \cdot \pi(C) \geq 0 \\ B > \pi(D) \qquad B \cdot \pi(C) > 0 \\ \pi^*(K_Z + B + J) > 0 \end{cases}$$

$\lightning$

$$H^i(\mathcal{O}_{\Delta_J}) \underset{\text{Hodge theory}}{=} H^0(\Omega_{\Delta_J}) = 0 \qquad \square$$

Upshot $\quad H^i(\mathcal{O}_\Delta) \simeq H^i(D(\Delta))$

Thm. $\quad H^i(D(\Delta), \mathbb{C}) = 0 \qquad 0 < i < \dim D(\Delta)$

Pf. $\quad \mathcal{O}_X(-\Delta) \to \mathcal{O}_X \to \mathcal{O}_\Delta$

$$0 = H^i(\mathcal{O}_X) \to H^i(\mathcal{O}_\Delta) = H^i(D(\Delta))$$

$$\to H^{i+1}(\mathcal{O}_X(-\Delta)) \to H^{i+1}(\mathcal{O}_X) = 0$$

X rat. conn. $\Rightarrow H^i(\mathcal{O}_X) = 0$
for $i > 0$

$$H^{i+1}(\mathcal{O}_X(-\Delta)) = H^{\dim X - i - 1}(K_X + \Delta)$$

$$= H^{\dim X - i - 1}(\mathcal{O}_X) = 0$$

$K_X + \Delta \sim 0$

if $\dim X - i - 1 > 0$

$\dim \mathcal{D}(\Delta) = \dim \Delta = \dim X - 1$

$\boxed{i = \dim \mathcal{D}(\Delta)}$ $\quad H^{\dim \mathcal{D}(\Delta)}(\mathcal{D}(\Delta))$

$$= H^0(K_X + \Delta),$$

the orientability of $\mathcal{D}(\Delta)$ measures the index of $K_X + \Delta$, i.e. when $K_X + \Delta \sim 0$

$\boxed{i = 0}$ $\quad$ Thm. [Kollár] if $\mathcal{D}(\Delta)$ is not connected, then $\mathcal{D}(\Delta) \simeq S^0$

Big question. Can we find an algebraic interpretation of Tors $H^*(\mathcal{D}(\Delta), \mathbb{Z})$?

In low dim it is controlled by the fundamental group.

<u>Guess</u>

$$(\mathcal{X}, \Delta)$$
$$\downarrow$$
$C = \operatorname{Spec} DVR$ mixed char

- $(\mathcal{X}_\eta, \Delta_\eta)$ dlt log CY pair in char 0
- $(\mathcal{X}_0, \Delta_0)$ ——— " ——— in char p
- good reduction : $D(\Delta_\eta) = D(\Delta_0)$

$H^i(D(\Delta), \mathbb{Z}/p\mathbb{Z})$ is a subobj of $H^i(\mathcal{O}_{\Delta_0})$

or

the residue field of C

$H^i(X, \mathcal{O}_X) = 0$ for X Fano variety in char p

<u>Ex</u>. [Kollár - Xu]

$X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with $[x_1, y_1], [x_2\, y_2][x_3 y_3]$

Δ_X = toric boundary $\quad (\mathbb{P}^1, 0+\infty) \times \ldots$

$\tau : [x_1 : y_1][x_2 : y_2][x_3 : y_3]$

$\longrightarrow [y_1 : x_1][y_2 : x_2][y_3 : x_3]$

$Y = X/\tau$

$\Delta_Y = q_* \Delta_X \qquad q : X \to X/\tau$

Claim (Y, Δ_Y) is dlt and log CY

$\underline{\text{Pf.}}$ Fix $\tau = \{[1 : \pm 1][1 : \pm 1], [1 : \pm 1]\}$

$= 8$ pts.

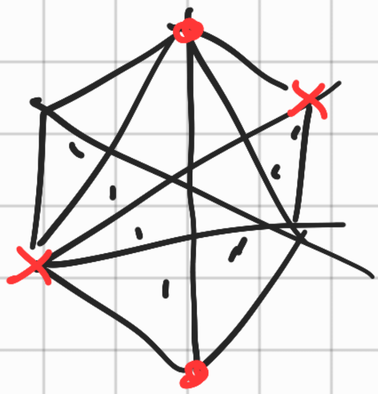
q is quasi-étale

$0 \sim K_X + \Delta_X = q^*(K_Y + \Delta_Y) \Rightarrow$ log CY

Fix τ is away from $\Delta_X \Rightarrow \Delta_Y$ is snc

$D(\Delta_Y) = D(\Delta_X)/\tau \simeq S^2/\pm 1 \simeq \mathbb{R}\mathbb{P}^2$

$\mathbb{R}^3$: fan of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$



τ acts as the antipodal involution

char 2. Y is a Fano compactification of an isolated singularity studied by Totaro terminal sing which is not CM in char 2.

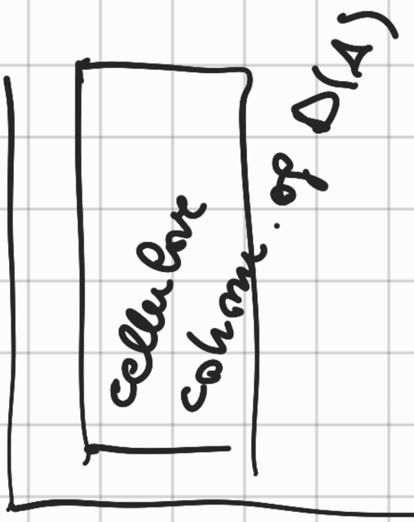
If CM

$$K_Y \sim \mathcal{O}(-\Delta) \qquad H^2(Y, K_Y) \simeq H^1(Y, \mathcal{O}_Y) = 0$$

$$H^2(Y, K_Y) = H^2(Y, \mathcal{O}(-\Delta_Y)) \neq 0$$

$$0 \neq H^1(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z}) = H^1(\mathcal{O}(\Delta_Y)) = H^1(\mathcal{O}_{\Delta_Y})$$

$$0 = H^1(\mathcal{O}_Y) \to H^1(\mathcal{O}_{\Delta_Y}) \to H^2(Y, \mathcal{O}(-\Delta)) \to H^2(\mathcal{O}_Y) = 0$$



$H^i(\mathcal{O}_{\Delta_J})$ in this case Δ_J is rational

Open question

$$\begin{array}{ccc} (X, \Delta_X) & \overset{K_X\text{-MMP}}{\dashrightarrow} & (Y, \Delta_Y) \\ & & \downarrow \text{Mori fiber space} \\ & & Z \end{array}$$

- relate the dual complex $\mathcal{D}(\Delta_X)$ with $\mathcal{D}(\Delta_Y)$
- identify special subcomplex (hor and vert) in $\mathcal{D}(\Delta_Y)$ and glue them together.

[Kollár - Xu] If $\dim \mathcal{D}(\Delta) \geq 2$, then

$$\pi_1(X^{\mathrm{sm}}) \twoheadrightarrow \pi_1(\mathcal{D}(\Delta))$$